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Some Properties of Lines in Space of Four Dimensions and their Interpretation in the Geometry of the Circle in Space of Three Dimensions.

By C. L. E. Moore.

The study of systems of lines in space of four dimensions is so closely related to the study of systems of circles and other geometric configurations that it has seemed worth while to develop this geometry and to interpret the results in the geometry of the circle. Klein[*] has suggested the study of lines in space of four dimensions as a possible approach to the study of systems of circles. The only extensive study made of systems of lines in S_4 is that of Castelnuovo;[†] but this is concerned only with linear systems of lines, and as linear systems of circles have already been studied by Koenigs[‡] and Cosserat,[§] no further reference will be made to linear systems of lines. The properties of lines treated in this paper involve the first derivatives.

Ruled Surfaces.

We will define a line in a space of four dimensions S_4 by means of the four equations

$$\left.\begin{aligned} x_1 &= a_1 + \alpha_1 z, \\ x_2 &= a_2 + \alpha_2 z, \\ x_3 &= a_3 + \alpha_3 z, \\ x_4 &= z; \end{aligned}\right\} \quad (1)$$

and to define a ruled surface we will consider $a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3$ as functions of a variable t. Throughout this paper the functions are supposed to be continuous and to possess first derivatives which are also finite and continuous.

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[*] "Einleitung in die Höhere Geometrie," Vol. I, p. 242.

[†] *Atti del reale Instituto Veneto*, 1890–91.

[‡] "Contributions à la théorie du cercle dans l'espace," *Annales de la Faculté des Sciences de Toulous*, t. II, 1888.

[§] "Sur la cercle considéré comme élément générateur de l'espace," *Annales de Toulous*, t. III, 1889.

The tangent plane at a point z, t of the ruled surface (1) (the coefficients being considered functions of t) is then

$$\begin{aligned} x_1 &= a_1 + \alpha_1 z + \tau \alpha_1 + t_1 (a_1' + z \alpha_1'), \\ x_2 &= a_2 + \alpha_2 z + \tau \alpha_2 + t_1 (a_2' + z \alpha_2'), \\ x_3 &= a_3 + \alpha_3 z + \tau \alpha_3 + t_1 (a_3' + z \alpha_3'), \\ x_4 &= z + \tau, \end{aligned}$$

where t_1, τ are the parameters and $a', \alpha', \ldots.$ denote $\frac{da}{dt}, \frac{d\alpha}{dt}, \ldots..$ If we express this plane as the intersection of two hyperplanes, we have

$$\begin{vmatrix} x_1 - a_1 - \alpha_1 z & x_2 - a_2 - \alpha_2 z & x_4 - z \\ \alpha_1 & \alpha_2 & 1 \\ a_1' + \alpha_1' z & a_2' + \alpha_2' z & 0 \end{vmatrix} = 0,$$

$$\begin{vmatrix} x_1 - a_1 - \alpha_1 z & x_3 - a_3 - \alpha_3 z & x_4 - z \\ \alpha_1 & \alpha_3 & 1 \\ a_1' + \alpha_1' z & a_3' + \alpha_3' z & 0 \end{vmatrix} = 0.$$

Developing these determinants, we have

$$\frac{x_1 - a_1 - \alpha_1 x_4}{x_2 - a \ - \alpha_2 x_4} = \frac{a_1' + \alpha_1' z}{a_2' + \alpha_2' z} = \rho_1, \tag{2}$$

$$\frac{x_1 - a_1 - \alpha_1 x_4}{x_3 - a_3 - \alpha_3 x_4} = \frac{a_1' + \alpha_1' z}{a_3' + \alpha_3' z} = \rho_2. \tag{3}$$

The first member of each of these equations is free from z, and therefore we see each of these hyperplanes depends linearly on z. Now given z, and the two hyperplanes, and consequently the tangent plane, is determined; and inversely, given ρ_1, then z is determined and consequently ρ_2. Hence, *there is a* (1, 1) *correspondence between the points of a generator r of a ruled surface and the planes tangent to the surface in these points;* that is, the Chasles correlation holds for ruled surfaces in S_4.*

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* From (2) and (3) we have, equating the values of z obtained from each,

$$(a_2 a_3 - a_3 a_2) \rho_1 \rho_2 + (a_1 a_2 - a_2 a_1) \rho_1 - (a_1 a_3 - a_3 a_1) \rho_2 = 0 \tag{4}$$

as the relation which must hold between ρ_1, ρ_2. If this relation holds, the locus of the plane [(2), (3)] is

$$(a_2 a_3 - a_3 a_2) \frac{x_1 - a_1 - a_1 x_4}{x_2 - a_2 - a_2 x_4} \cdot \frac{x_1 - a_1 - a_1 x_4}{x_3 - a_3 - a_3 x_4} + (a_1 a_2 - a_2 a_1) \frac{x_1 - a_1 - a_1 x_4}{x_2 - a_2 - a_2 x_4} + (a_1 a_3 - a_3 a_1) \frac{x_1 - a_1 - a_1 x_4}{x_3 - a_3 - a_3 x_4} = 0,$$

which is the equation of the hyperplane generated by the tangent planes to the ruled surface in points of a generator r.

We may consider a_1', a_2', a_3', α_1', α_2', α_3' as the homogeneous coordinates of this projectivity (correlation). If

$$a_1'\alpha' - a_2'\alpha_1' = 0, \tag{5}$$
$$a_1'\alpha_3' - a_3'\alpha_1' = 0, \tag{6}$$

the hyperplanes (2) and (3) are the same for every value of z; that is, for every point of the generator r. Hence the surface has the same tangent plane the whole length of the generator r. Equations (5) and (6) are then the conditions in order that the correlation be degenerate. When this condition is given we easily obtain the tangent plane from equations (2) and (3). We will call such a plane a *focal plane* or *singular plane.*

If we solve (2) and (3) for z, we have

$$z = -\frac{a_1' - a_2'\rho_1}{\alpha_1' - \alpha_2'\rho_1}, \tag{2'}$$
$$z = -\frac{\alpha_1' - a_3'\rho_2}{\alpha_1' - \alpha_3'\rho_2} \tag{3'}$$

(remembering that ρ_1 and ρ_2 are not independent). We see that (5) and (6) are also the conditions in order that z be independent of ρ_1 and ρ_2. In this case to the point z of r corresponds each plane which passes through r. Such a point we will call a *focus* or *singular point.* It is easily seen also that (5) and (6) are the conditions in order that the generator infinitely close to r intersect r, for the line infinitely close to r has for equations

$$\begin{aligned} x_1 &= a_1 + \alpha_1 z + d a_1 + z d\alpha_1, \\ x_2 &= a_2 + \alpha_2 z + d a_2 + z d\alpha_2, \\ x_3 &= a_3 + \alpha_3 z + d a_3 + z d\alpha_3, \\ x_4 &= z. \end{aligned}$$

Subtracting equations (1), we have

$$\begin{aligned} d a_1 + z d\alpha_1 &= 0, \\ d a_2 + z d\alpha_2 &= 0, \\ d a_3 + z d\alpha_3 &= 0, \end{aligned}$$

and (5) and (6) are at once seen to be the conditions in order that these equations be consistent.

Now if we consider a_1', a_2', a_3', α_1', α_2', α_3' as the homogeneous coordinates of a space Σ of five dimensions, each point of Σ will represent a correlation between the points of r and a singly infinite number of planes passing through r

and generating an ordinary space S_3. In this space Σ the intersection of the two cones whose equations are (5) and (6) represent the degenerate correlations. The intersection of these two cones is a variety V_3^3 of three dimensions and of the third order; since the S_3 $a_1' = \alpha_1' = 0$ is common to both cones, it is evident that the points of this S_3 do not represent degenerate correlations. We will designate this variety V_3^3 by Φ.

From equations (5) and (6) we see that the planes defined by

$$\left.\begin{aligned} a_1' + z\alpha_1' &= 0, \\ a_2' + z\alpha_2' &= 0, \\ a_3' + z\alpha_3' &= 0 \end{aligned}\right\} \qquad (7)$$

stand in the variety Φ for all values of z and represent the points of the generator r; that is, the points of r considered as *foci* are represented by (7). We also see that the lines represented by

$$\left.\begin{aligned} a_1' - \rho_1 a_2' &= 0, \\ \alpha_1' - \rho_1 \alpha_2' &= 0, \\ a_1' - a_3' \rho_2 &= 0, \\ \alpha_1' - \alpha_3' \rho_2 &= 0 \end{aligned}\right\} \qquad (8)$$

stand in Φ for all values of ρ_1 and ρ_2 and represent the planes passing through r; that is, the planes passing through r and considered as focal planes are represented by the lines (8). The lines of (8) which represent planes tangent to the the same ruled surface are given by the values of ρ_1 and ρ_2 which satisfy (4). Evidently each line represented by (8) will cut each plane represented by (7).

Congruences of Lines in S_4, C_2.

If a_1, a_2, a_3, α_1, α_2, α_3 are functions of two parameters u_1, u_2, equations (1) will generate a congruence of lines in S_4. Then the correlations which belong to a line r of the congruence, that is the correlations on r defined by all the ruled surfaces of the congruence which have r for generator, are represented in Σ by

$$(l) \quad \left\{\begin{aligned} a_1' &= \frac{\partial a_1}{\partial u_1}\,\frac{du_1}{dt} + \frac{\partial a_1}{\partial u_2}\,\frac{du_2}{dt}, \\ a_2' &= \frac{\partial a_2}{\partial u_1}\,\frac{du_1}{dt} + \frac{\partial \alpha_2}{\partial u_2}\,\frac{du_2}{dt}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ \alpha_3 &= \frac{\partial \alpha_3}{\partial u_1}\,\frac{du_1}{dt} + \frac{\partial \alpha_3}{\partial u_2}\,\frac{du_2}{dt}. \end{aligned}\right.$$

[We have here supposed that u_1 and u_2 are both functions of a parameter t; then by varying these functions we get all the surfaces of the congruence which contain the line r.] The above equations show that the correlations in a congruence C_2 which belong to a line r of the congruence are defined in Σ by a line (l). $\left(\text{The parameters of the line are } \frac{du_1}{dt}, \frac{du_2}{dt}.\right)$ A line in Σ will not, in general, cut the variety Φ. Hence: *In general, there are no developables belonging to a congruence of lines in a space of four dimensions* S_4; that is, there are no degenerate projectivities.

Particular Cases. 1. The line (l) may intersect Φ in one point. In this case there is one focus on r and one focal plane passing through r (since the point in which (l) cuts Φ must lie on a plane of (7) and a line of (8)). If this condition is fulfilled for each line of the congruence, *the focal points generate a focal surface, but the focal planes in general do not have an envelope.* The congruence is formed by the tangents to a system of lines traced on a surface.

2. The line (l) may cut Φ in two points, in which case there are two foci on r and two focal planes passing through r. If this condition is fulfilled for every generator r, *the congruence will have two focal surfaces, and through each line will pass two developables belonging to the congruence.* Let M, N be the two foci on any generator r, and let μ, ν be the corresponding focal planes. The plane ν is tangent to the locus of M. For any line of the congruence is tangent to both developables, and therefore tangent to both the locus of M and the locus of N. Now as r moves, generating one of the developables whose cuspidal edge is on the locus of N, it is always tangent to the surface generated by M; hence, the tangent planes to this developable are tangent to the locus of M, but these tangent planes are the planes ν. *This congruence is formed by the lines tangent to a system of characteristics*[*] *traced on a surface in* S_4. Every surface in S_4 can be the focal surface for such a congruence.

3. The two points in which the line (l) cuts Φ may coincide, in which case the two focal surfaces coincide and *the congruence is generated by the tangent lines to a system of characteristics traced on a surface of the parabolic type.*[†]

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[*] See Segre: "Su una classe di superficii degli iperspazii, ecc.," *Atti di Torino*, 1907. In this paper Segre has discussed a class of surfaces in S on which can be traced two systems of curves such that the tangent planes to the surface, in points of one of these curves, form a developable. Such curves he has called characteristics. It is shown that every surface in S_4 belongs to this class.

[†] A surface is said to be of the parabolic type when the two systems of characteristics coincide. In this case the coordinates of a point of the surface satisfy a partial differential equation of the second order and of the parabolic type.

Specializations of the Focal Surface. In general the foci in cases 1, 2, 3 above generate surfaces, but they may generate curves in particular cases. In case 1, if the focal surface becomes a curve, the congruence will consist of the cones of lines whose vertices lie on the curve; but here again these cones have no envelope.

If one of the focal surfaces in case 2 is a curve, the second surface can not be a general surface but must be the envelope of ∞^1 cones whose vertices lie on the curve. This requires that the one system of characteristics should be such that the tangent planes to the surface along points of one of these characteristics should form a cone, and in general this will not be the case. The first system of focal planes consists of the tangent planes of the cones, and the second system consists of the planes passing through the lines of the congruence and tangent to the curve. The focal points on any line are the points where it cuts the curve and is tangent to the surface.

If both focal surfaces become curves, then the developables become the cones of lines whose vertices are on one curve and which touch the other curve. If the two curves are straight lines, the congruence becomes a congruence in S_3.

The two curves may coincide, and, as in S_3, the congruence consists of the pencils of lines whose planes are tangent to the curve and whose vertices lie on the curve.

Other specializations may occur, but in general they occur only for particular lines of the congruence and not identically. It may happen that the line (l) lies wholly within the variety Φ. This may occur in two ways: (a) it may lie in one of the planes (7), in which case there is one focus corresponding to this plane, but there are ∞^1 focal planes corresponding to the lines of (8) which are cut by the line (l); that is, *there is one focus on the line r, but ∞^1 planes passing through r are focal planes.* These focal planes generate an ordinary space S_3; that is, they form a pencil with r for axis. For if we consider any hyperplane in Σ, it will cut any given plane of (7) in a line s; and it is easily seen that the relation which holds between ρ_1 and ρ_2 for the lines of (8) which cut s is a bilinear relation without a constant term, and this by equations (2) and (3) determines a pencil of planes. (b) The line (l) may coincide with a line of (8), *in which case each point of r is a focus but only one plane is a focal plane.*

Examples of lines of the kind (a) would be the lines tangent to the curve of intersection of the two focal surfaces (if such exist). In this case we see that evidently the two foci have coincided in the point of tangency, but the two planes passing through r and tangent to the two focal surfaces are the two focal planes;

but we saw that if the foci coincide, the focal planes must coincide unless the line (l) lies in Φ. Hence ∞^1 planes of the pencil determined by these two focal planes are focal planes. Lines of the class (b) would be illustrated by the lines of the developable (if such exist) which envelopes the two sheets of the focal surface. Evidently the two points of contact of such a line r with the two sheets of the focal surface are foci; but the two focal planes have coincided, and as the focal planes can not coincide if the focal points do not coincide, unless the line (l) lies in Φ, and as there are two foci, there must be an infinite number.

Triply Infinite Systems of Lines, C_3.

If the coefficients $a_1, \ldots, \alpha_1, \ldots$ are functions of three variables u_1, u_2, u_3, the coordinates of the correlation become

$$(\Pi)\quad \left\{ \begin{array}{l} a_1' = \dfrac{\partial a_1}{\partial u_1}\dfrac{du_1}{dt} + \dfrac{\partial a_1}{\partial u_2}\dfrac{du_2}{dt} + \dfrac{\partial a_1}{\partial u_3}\dfrac{du_3}{dt}, \\ a_2' = \dfrac{\partial a_2}{\partial u_1}\dfrac{du_1}{dt} + \dfrac{\partial a_2}{\partial u_2}\dfrac{du_2}{dt} + \dfrac{\partial a_2}{\partial u_3}\dfrac{du_3}{dt}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \alpha_3' = \dfrac{\partial \alpha_3}{\partial u_1}\dfrac{du_1}{dt} + \dfrac{\partial \alpha_3}{\partial u_2}\dfrac{du_2}{dt} + \dfrac{\partial \alpha_3}{\partial u_3}\dfrac{du_3}{dt}. \end{array} \right.$$

Hence the correlations which belong to any line r of the complex are represented in Σ by a plane (Π).

The plane Π will cut the variety Φ in three points; hence, *through each line of the triply infinite system pass three developables.*[*] The system of three parameters then consists of three systems of ∞^2 developables. The focal points generate three focal varieties of three dimensions ϕ_3', ϕ_3'', ϕ_3'''. These varieties may be considered as the locus of the edges of regression of the three systems of developables. If we consider one of the developables whose edge of regression is on ϕ_3', for example, all the lines of the developable are tangent to ϕ_3'', ϕ_3''', and therefore the planes of these developables are tangent to ϕ_3'' and ϕ_3'''. Therefore, *if $M_1\mu_1$, $M_2\mu_2$, $M_3\mu_3$ are the foci and corresponding focal planes of a line r, the plane μ_1, for example, is tangent to the loci generated by M_2 and M_3.*

The system can be generated as the intersection of three special systems of ∞^5 lines formed by tangents to three varieties of three dimensions; *i. e.*, three hypersurfaces.

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[*] In the *Rendi Conti di Palermo*, 1890, Segre has demonstrated the general theorem that *through each line of a system of lines depending on $n-1$ parameters in a space of n dimensions will pass $n-1$ developables, and each line of the system touches $n-1$ focal varieties of $n-1$ dimensions.* Following the procedure above, this general theorem will be demonstrated later.

Either two or three of the points in which Π cuts Φ may coincide. In the first case ϕ_3', ϕ_3'' coincide, say; then all the focal planes are tangent to ϕ_3', but only the planes μ_1 are tangent to ϕ_3'''. If all three points in which Π cuts Φ coincide, there is only one focal variety, and the focal planes are all tangent to this variety. This system is formed by the tangent lines to ∞^2 curves traced on a hypersurface whose osculating planes are tangent planes to the hypersurface.

Singular Cases. 1. The plane Π may coincide with one of the planes (7), in which case *on r there is one focus, but each of the ∞^2 planes passing through r is a focal plane;* or

2. The plane Π may cut Φ in a line of (8), in which case *there is one focal plane, but every point of r is a focal point;* or

3. Π may cut one of the planes of (7) in a line, in which case *one point of r is a focus, but ∞^1 planes of a pencil are focal planes;* or

4. Π may cut one of the planes of (7) in a line and contain one of the lines of (8). In this case *there is one focus, but ∞^1 focal planes* (corresponding to the lines of (8) which cut the line in which Π cuts the plane of (7)); *but corresponding to the line of* (8) *we have one focal plane and ∞^1 focal points.* Thus we see that the plane and point corresponding to the lines in which Π cuts Φ play a sort of double rôle.

5. The plane Π may cut Φ in a conic not lying in a plane of (7); for an S_3 which contains two lines of (8) would intersect Φ in a quadric surface, and hence, if Π lies in such an S_3, it would cut Φ in a conic. In this case *each point of r is a focus and ∞^1 planes passing through r are focal planes.* This differs from the preceding case, because there is a (1, 1) correspondence between the foci and focal planes; that is, to each focus is associated a definite focal plane. The focal planes form a plane pencil. An example of this is furnished by a complex of lines in a space of three dimensions.

The above conditions are not, in general, satisfied identically, but occur only for particular lines of the system. An example of lines of class 1 is the complex of lines through a point. In this complex each line is of class 1; that is, the relation is an identity. Another example is furnished by the lines tangent to the curve of intersection of ϕ_3', ϕ_3'', ϕ_3'''. The point of tangency is the point of contact, and the focal planes four in number are the three planes tangent to the three varieties ϕ_3', ϕ_3'', ϕ_3''' and the osculating plane of the curve. But we saw that there can be only three focal planes unless Π cuts Φ in a particular manner, in which case there must be an infinite number of focal planes. There is one essential difference between these two examples. In the first, through each line pass an infinite number of developables, while in the second, only four. In this second case, strictly speaking, the lines tangent to the curve should be counted as a part of the envelope of the focal planes, but following the custom of congruences of lines in S_3, we shall not include it here.

If the three varieties ϕ_3', ϕ_3'', ϕ_3''' have a common developable of tangent planes, then the generators of this developable are examples of lines of class 2. On each line there are four foci, the three points of contact with the three varieties and the point on the edge of regression of the developable. But if more than three points of r are foci, then all the points are foci. In this case also, strictly speaking, this developable should be counted as a part of the locus of the focal points, but here again we shall not consider it as such.

An example of a system of class 3 is furnished by the system of lines tangent to a surface. In this case the osculating planes to the curves which are traced on the surface, and which have a given tangent line r, form the focal planes of the system. These osculating planes form a plane pencil.*

Specializations. One of the focal hypersurfaces may become a surface. The system then consists of all the lines which cut a surface and are tangent to two hypersurfaces. The developables corresponding to the *focal surface* consist of the cone of lines whose vertex is on the surface and tangent to the two focal hypersurfaces. In the same manner two or three of the focal hypersurfaces may become surfaces. If one of the focal varieties reduces to a curve, it is at once seen that two of the focal varieties must coincide in this curve. The system then consists of all the lines which cut a curve and either are tangent to a hypersurface or cut a surface.

Systems of ∞^4 *Lines,* C_4.

If the coefficients $a_1, a_2, \ldots.$ are functions of four parameters u_1, u_2, u_3, u_4, the coordinates of the correlation are

$$(U)\quad \left\{\begin{aligned}
a_1' &= \frac{\partial a_1}{\partial u_1}\frac{du_1}{dt} + \frac{\partial a_1}{\partial u_2}\frac{du_2}{dt} + \frac{\partial a_1}{\partial u_3}\frac{du_3}{dt} + \frac{\partial a_1}{\partial u_4}\frac{du_4}{dt},\\
a_2' &= \frac{\partial a_2}{\partial u_1}\frac{du_1}{dt} + \frac{\partial a_2}{\partial u_2}\frac{du_2}{dt} + \frac{\partial a_2}{\partial u_3}\frac{du_3}{dt} + \frac{\partial a_2}{\partial u_4}\frac{du_4}{dt},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\alpha_3' &= \frac{\partial \alpha_3}{\partial u_1}\frac{du_1}{dt} + \frac{\partial \alpha_3}{\partial u_2}\frac{du_2}{dt} + \frac{\partial \alpha_3}{\partial u_3}\frac{du_3}{dt} + \frac{\partial \alpha_3}{\partial u_4}\frac{du_4}{dt}.
\end{aligned}\right.$$

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* See Segre: "Su una classe di superficii, ecc.," p. 7.

We have seen that the points of the line r are represented by the planes of the system (7) and that the planes passing through r are represented by the lines of the system (8). Every Chasles correlation establishes a (1, 1) correspondence between the planes of (7) and ∞^1 lines of (8), but evidently not every correspondence of this sort represents such a correlation. In the first place it would be necessary for the ∞^1 planes through r which correspond to the ∞^1 lines of (8) to generate an S_3. Any S_3 in Σ cuts the variety Φ in a cubic curve, and thereby a (1, 1) correspondence is established between the points of r and ∞^1 planes passing through r; but not every such correspondence represents a Chasles correlation, for in Σ there are ∞^8 spaces S_3, but there are only ∞^5 Chasles correlations. In order to determine which of these cubics represent a Chasles correlation, we will first find the condition in order that two correlations be in involution.

By equation (2) a projectivity is established between the hyperplanes ρ_1 and the points z. Equation (5) is the condition in order that this projectivity should degenerate; hence, in order that two projectivities $(a_1', a_2', a_3', \alpha_1', \alpha_2', \alpha_3')$ and $(\bar{a}_1', \bar{a}_2', \bar{a}_3', \bar{\alpha}_1', \alpha_2', \alpha_3')$ be in involution the points in the space Σ represented by them must be conjugate with respect to the quadric (5). The same reasoning will show that the points should also be conjugate with respect to the quadric (6). Therefore the two projectivities are in involution when

$$a_1'\bar{\alpha}_2' + \bar{a}_1'\alpha_2' - a_2'\alpha_1' - \bar{a}_2'\alpha_1' = 0, \tag{9}$$

$$a_1'\bar{\alpha}_3' + \bar{a}_1'\alpha_3' - a_3'\bar{\alpha}_1' - \bar{a}_3'\alpha_1' = 0. \tag{10}$$

Then all the projectivities which are in involution with a given projectivity lie in the S_3 formed by the intersection of (9) and (10). This S_3 cuts Φ in a cubic, and the points of this cubic represent the degenerate projectivities which are in involution with the two given ones; but in order that a non-degenerate projectivity should be in involution with a degenerate one, it is necessary that the singular point and the singular plane of the degenerate projectivity should be corresponding elements of the non-degenerate one. Therefore we see that the correlation corresponding to any point of Σ is the same as that which is set up by the points of the cubic of intersection of Φ with the polars of the point with respect to the two quadrics (5) and (6). Hence we have the theorem: *The cubics which determine a Chasles correlation are the intersections of* Φ *with the polars of the points of* Σ *with respect to the quadrics* (5) *and* (6).

The space U which represents the correlations belonging to C_4 cuts Φ in a cubic, and therefore a (1, 1) correspondence is established between the points of r and ∞^1 planes passing through r; but as we saw, these planes do not, in general, generate an S_3. To find the locus of these planes, we need only find the relation established between ρ_1 and ρ_2, and then from equations (2) and (3) find the equation of the locus. Let the S_3 be given by the equations

$$\Sigma A_i a_i' = 0, \qquad \Sigma B_i a_i' = 0.$$

Then the condition that this S_3 cut a line of (8) is

$$(B) \qquad \begin{vmatrix} 1 & -\rho_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\rho_1 & 0 \\ 1 & 0 & -\rho_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\rho_2 \\ A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{vmatrix} = 0.$$

This is the relation sought between ρ_1 and ρ_2. It is easily seen from this determinant that there is no constant term or terms of the first degree in ρ_1 and ρ_2. The determinant when expanded is of the form

$$(C) \qquad \rho_1^2\rho_2^2 + a\rho_1^2\rho_2 + b\rho_1\rho_2^2 + c\rho_1\rho_2 + d\rho_1^2 + e\rho_2^2 = 0,$$

and substituting the values of ρ_1, ρ_2 from equations (2) and (3), we have the cone

$$x^2 + axz + bxy + cyz + dz^2 + ey^2 = 0,$$

where

$$x = x_1 - a_1 - \alpha_1 x_4,$$
$$y = x_2 - a_2 - \alpha_2 x_4,$$
$$z = x_3 - a_3 - \alpha_3 x_4.$$

The condition that this cone should factor is the vanishing of the discriminant. Hence, *in* C_4 *there are* ∞^3 *lines whose cone of singular planes factors into two ordinary spaces* S_3. These two spaces have a plane Π in common.

If the space U is the conjugate of a point with respect to Φ, that is, if it is of the kind described above, the (1, 1) correspondence becomes a Chasles correlation; this, however, imposes three conditions. Hence, *in* C_4 *there are* ∞^1 *lines (i. e., a ruled surface) such that the* (1, 1) *correspondence established by* U *between the points of* r *and* ∞^1 *planes passing through* r, *becomes a correlation of Chasles.*

Singular Cases. 1. The space U may be tangent to Φ, in which case the intersection will evidently consist of the plane and line of Φ which pass through the point of tangency. The space U is evidently tangent to (5) and (6) in the same point, and consequently the curve of intersection represents a Chasles correlation corresponding to the point of contact. Hence, *there is one point and one plane which belongs to each correlation on r* (this point and plane correspond to the plane and line in which U cuts Φ). For consider any point Q' of U; its polar with respect to (5) and (6) will evidently pass through the point of contact Q of U, and consequently Q represents a point of r and its corresponding focal plane. Hence, *all the ruled surfaces which pass regularly through such a line r are tangent in this point and have this plane for tangent plane.*

If U is tangent to Φ, five conditions are imposed; hence, in C_4 there are, in general, no lines of the kind mentioned above.

2. (a) U may cut Φ in a line of (8). The cubic of intersection then degenerates into this line and a conic, in which case the (1, 1) correspondence between the points of r and ∞^1 planes passing through r degenerates into two parts: (1) The line cuts each plane of (7); hence, there is one focal plane, but each point of r is a focus corresponding. (2) The correspondence set up by the conic is still (1, 1). In this case the focal planes passing through r generate an S_3; for in the determinant (B) all the minors of order 5 must vanish, in particular the minor formed by canceling the second row and fifth column, which gives the following relation:

$$a\rho_1\rho_2 + b\rho_1 + c\rho_2 = 0;$$

and hence the locus of the planes is an S_3.

(b) The space U may cut Φ in two lines of (8); then it will cut Φ in a regulus. In this case the correlations which belong to any triply infinite system contained in C_4 are represented by a plane which cuts Φ in a conic, and the pole of the plane with respect to the quadric will be a correlation in involution with all the correlations belonging to the triply infinite system. Within a system like this there is a surface of singularities, and the whole behavior is like that of the lines in an S_3.

Fivefold Infinite Families of Lines, C_5.

The coefficients are now functions of five parameters, and therefore the correlations contained in C_5 are represented in Σ by an S_4 cutting the hypersurface Φ in a cubic surface standing in the S_4. Every cubic surface in a space of four dimensions must be a ruled surface having a rectilinear directrix. In this case the directrix must necessarily be a line of (8). For the lines of Φ which cut a line lying in one of the planes (7) will generate a quadric, because the lines of (8) set up a (1, 1) correspondence between the planes of (7), and therefore to a line in one plane will correspond a line in any other plane, and the lines joining corresponding points of these two lines will generate a quadric. The S_4 cuts each plane of (7) in a line; and therefore each point of r is a focus, and corresponding to any point there are ∞^1 planes passing through r which are focal planes. These planes correspond to the lines of (8) which cut the line in which S_4 cuts the plane of (7) which corresponds to the given point. The planes which have any given point for focus generate an S_3, and all these spaces S_3 have a plane in common (corresponding to the directrix of the cubic surface). Between the points of r and these spaces S_3 there is then a (1, 1) correspondence. This correspondence may degenerate; then the spaces S_3 coincide. This would necessitate that the lines which cut any line which S_4 cuts from a plane of (7) would cut the lines which S_4 cuts from the other planes. If this happens, then it is evident that the cubic surface will degenerate. S_4 is then tangent to Φ. The surface then degenerates into a quadric and a plane of (7); then there is one point of r (corresponding to the plane just mentioned) to which every plane through r corresponds. Also, when we consider the quadric, we see that each point of r has the same ∞^1 planes for corresponding focal planes. In any system C_5 there is a C_4 of lines of this kind, since the condition that S_4 be tangent to Φ imposes one condition. We may designate the lines of this C_4 by the term *singular lines*.

The lines of C_5 which pass through a given point generate a cone of three dimensions. The planes tangent to this cone in any given line r form an ordinary pencil with r as axis (*i. e.*, generate a space S_3), and the cones whose vertices all lie on a line r of the system have one common tangent plane. If the line r is such that the space S_4 is tangent to Φ (*i. e.*, if r is a singular line), then the cones whose vertices lie on this line have an infinite number of common tangent planes. There are ∞^4 such lines. In the general case a plane tangent to the cone is a plane which contains two infinitely near lines of the system C_5; hence, from the above we see that *through each line r of C_5 there passes one plane which contains ∞^1 lines of the system infinitely near to r.*

In any hyperplane S_3 of S_4 there are ∞^3 lines which belong to any C_5; that is, an S_3 cuts an ordinary complex of lines from C_5.

Generalization.

If we consider a ruled surface in a space of n dimensions, and use equations analogous to (1), the equations which define the correlation between the points of a generator r and the tangent planes of the surface passing through r are of the form:

$$
\begin{array}{ll}
(1) & \dfrac{x_1 - a_1 - \alpha_1 x_n}{x_2 - a_2 - \alpha_2 x_n} = \dfrac{a'_1 + \alpha'_1 z}{a'_2 + \alpha'_2 z} = \rho_1, \\
(2) & \dfrac{x_1 - a_1 - \alpha_1 x_n}{x_3 - a_3 - \alpha_3 x_n} = \dfrac{a'_1 + \alpha'_1 z}{a'_3 + \alpha'_3 z} = \rho_2, \\
 & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\
 & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\
(n-2) & \dfrac{x_1 - a_1 - \alpha_1 x_n}{x_{n-1} - a_{n-1} - \alpha_{n-1} x_n} = \dfrac{a'_1 + \alpha'_1 z}{a'_{n-1} + \alpha'_{n-1} z} = \rho_{n-2}.
\end{array}
$$

The coordinates of the correlation are

$$a'_1,\ a'_2,\ a'_3,\ \dots,\ a'_{n-2},\ a'_{n-1},\ \alpha'_1,\ \alpha'_2,\ \alpha'_3,\ \dots,\ \alpha'_{n-2},\ \alpha'_{n-1},$$

where $a'_1 = \dfrac{da_1}{dt}$, $\dots, a_1, a_2, \dots, \alpha_1, \alpha_2, \dots$ all being assumed to be functions of the parameter t. Then

$$
(A)\ \left\{
\begin{array}{l}
a'_1 \alpha'_2 - a'_2 \alpha'_1 = 0, \\
a'_1 \alpha'_3 - a'_3 \alpha'_1 = 0, \\
\dots\dots\dots\dots\dots, \\
\dots\dots\dots\dots\dots, \\
a'_1 \alpha'_{n-1} - a'_{n-1} \alpha'_1 = 0
\end{array}
\right.
$$

are the conditions that the correlation should degenerate. Then, if we consider $a'_1, a'_2, \dots, a'_{n-1}, \alpha'_1, \dots, \alpha'_{n-1}$ as the homogeneous coordinates of a point in space of $2n-3$ dimensions, the degenerate correlations are represented by the variety (A) of $n-1$ dimensions and of order $n-1$, since the space $a_1 = \alpha_1 = 0$ is common to all the cones (A). The points of a generator r of the ruled surface are represented by the linear spaces of $n-2$ dimensions:

$$
\begin{array}{l}
a'_1 + \alpha'_1 z = 0, \\
a'_2 + \alpha'_2 z = 0, \\
a'_3 + \alpha'_3 z = 0, \\
\dots\dots\dots\dots, \\
\dots\dots\dots\dots, \\
a'_{n-1} + \alpha'_{n-1} z = 0,
\end{array}
$$

where z is the parameter of the point on the generator. The planes passing through r are represented by the lines:

$$
\begin{aligned}
a'_1 - \rho_1 a'_2 &= 0, \\
\alpha'_1 - \rho_1 \alpha'_2 &= 0, \\
a'_1 - \rho_2 a'_3 &= 0, \\
\alpha'_1 - \rho_2 \alpha'_3 &= 0, \\
\cdots\cdots\cdots\cdots &, \\
\cdots\cdots\cdots\cdots &, \\
a'_1 - \rho_{n-2} a'_{n-1} &= 0, \\
\alpha'_1 - \rho_{n-2} \alpha'_{n-1} &= 0.
\end{aligned}
$$

Now, for the properties of systems of lines in a space of n dimensions we can study the intersection of the variety (A) with the space which represents the correlations contained within the given system, just as in the preceding pages.

If we have an $(n-1)$-fold infinite system (that is, if $a_1, a_2, \ldots, a_{n-1}$, $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are functions of $n-1$ parameters), then

$$
\begin{aligned}
a'_1 &= \frac{\partial a_1}{\partial u_1}\frac{du_1}{dt} + \frac{\partial a_1}{\partial u_2}\frac{du_2}{dt} + \cdots + \frac{\partial a_1}{\partial u_{n-1}}\frac{du_{n-1}}{dt}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
\alpha'_{n-1} &= \frac{\partial \alpha_{n-1}}{\partial u_1}\frac{du_1}{dt} + \frac{\partial \alpha_{n-1}}{\partial u_2}\frac{du_2}{dt} + \cdots + \frac{\partial \alpha_{n-1}}{\partial u_{n-1}}\frac{du_{n-1}}{dt}.
\end{aligned}
$$

The correlations contained in this system are therefore represented by a space of $n-1$ dimensions. Now, a space of $n-1$ dimensions will intersect the variety (A) in $n-1$ points. Hence, we have the theorem due to Segre (*Rendi Conti di Palermo,* 1890): *In S_n an $(n-1)$-fold infinite system of lines has $n-1$ focal hypersurfaces.* Using the same reasoning as in S_4, we see that each focal plane is tangent to all the focal hypersurfaces except that one which contains the focus corresponding to this plane.

Application to the Geometry of the Circle in Space.

If the equation of the sphere be written in the form

$$
t(x^2 + y^2 + z^2) - 2ax - 2by - 2cz + 2d = 0, \tag{11}
$$

the ratio of the five quantities t, a, b, c, d will determine any sphere in space. The spheres of ordinary space can then be represented by points in a space of four dimensions. The radius of (11) is

$$
\rho^2 = \frac{a^2 + b^2 + c^2 - 2dt}{t^2},
$$

from which we see that the point spheres (spheres of zero radius) are represented by a quadric in S_4,

$$\phi = a^2 + b^2 + c^2 - 2dt = 0. \tag{12}$$

The planes of ordinary spaces are represented by the hyperplane

$$t = 0.$$

This hyperplane is tangent to ϕ in the point $(0, 0, 0, 0, 1) \equiv \omega$ which corresponds to the plane at infinity in ordinary space. The lines in the plane at infinity in ordinary space correspond in S_4 to the lines in the hyperplane $t = 0$ which pass through the point ω. The hyperplane $t = 0$ intersects the quadric ϕ in an ordinary quadric cone K, vertex ω, the generators of which represent the lines in the plane at infinity tangent to the circle at infinity. The other minimum lines in S_3 are represented in S_4 by tangent lines to this cone K.

The angle between two spheres (t, a, b, c, d) and (t', a', b', c', d') is given by the relation

$$\cos\theta = \frac{aa' + bb' + cc' - dt' - d't}{\sqrt{a^2 + b^2 + c^2 - 2dt}\ \sqrt{a'^2 + b'^2 + c'^2 - 2d't'}}. \tag{14}$$

In S_4 this represents the anharmonic ratio between the two points (t, a, b, c, d), (t', a', b', c', d') and the two points in which the line joining them cuts the quadric ϕ.

If the spheres are orthogonal,

$$aa' + bb' + cc' - dt' - d't = 0; \tag{15}$$

and hence we see that the equation

$$au_1 + bu_2 + cu_3 + du_4 + tu_5 = 0 \tag{16}$$

represents all the spheres which cut the fixed sphere $(-t, a, b, c, -d)$ orthogonally. Then in S_4 the hyperplane represents all the spheres orthogonal to a given sphere.

It is seen immediately from (11) that the spheres whose coordinates are

$$(\lambda t + \mu t', \quad \lambda a + \mu a', \quad \lambda b + \mu b', \quad \lambda c + \mu c', \quad \lambda d + \mu d')$$

are the spheres of the pencil determined by the two spheres (t, a, b, c, d), (t', a', b', c', d'); or, if we consider the envelope of these spheres, we may say that this represents a circle. But in S_4 this represents a line. Hence, the circles in ordinary space are represented by lines in S_4. If we consider the line as defined by the two points in which it cuts ϕ, we have the representation of circles due to Laguerre.*

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* *Œuvres*, Vol. II, p. 54.

Two spheres are tangent if the line which represents their circle of intersection is tangent to ϕ. The circle determined by two tangent spheres is a point circle. Therefore the point circles of S_3 correspond to the lines of S_4 which are tangent to ϕ. In particular, the lines which lie in ϕ represent those circles the spheres of whose pencil are all point spheres. Therefore, *the lines of* ϕ *correspond to the point circles composed to two coincident minimum lines.*

From (16) it was seen that the spheres of a hyperplane represent spheres orthogonal to a fixed sphere (represented by the pole of the hyperplane with respect to ϕ). But if two spheres are orthogonal to a fixed sphere, their circle of intersection is also orthogonal to this sphere. Then the circles of S_3 which are orthogonal to a fixed sphere are represented in S_4 by the lines in an S_3. Thus we have at once the relation between the ordinary line geometry and the geometry of circles orthogonal to a given sphere.*

Circles which lie on the same sphere (*i. e.*, which intersect in two points) are represented by lines which intersect in S_4. Annular surfaces and circled surfaces in S_3 are represented in S_4 by developable and ruled surfaces respectively.

In S_4 the plane is the intersection of two hyperplanes; hence a plane represents the circles which pass through two fixed points; viz., the points in S_3 represented by the points of contact of the two hyperplanes passing through the given plane and tangent to ϕ.

If a (1, 1) correspondence is established between the spheres of two pencils, the circles of intersection of corresponding spheres will generate a cyclide. Then in S_4 a quadric surface corresponds to a cyclide in ordinary space. Circled surfaces having two circle directrices which are cut by each circle in two points, are represented in S_4 by ruled surfaces having two right-line directrices, and therefore which stand in an S_3. Hence, such surfaces are composed of circles orthogonal to a fixed sphere.†

In this discussion we shall limit ourselves to configurations which properly stand in an S_4 and not in an S_3, since the latter are disposed of in the paper referred to.

All circles which are tangent to a fixed line in a fixed point are represented in S_4 by lines of a plane which is tangent to ϕ. Two circles which intersect in a single point are represented by lines in S_4 which determine an S_3 tangent to ϕ.

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* Moore: *Annals of Mathematics*, Series 2, Vol. VIII, p. 57. Forbes: "The Geometry of Circles Orthogonal to a Given Sphere," New York, 1904.

† See Moore: *Annals of Mathematics*, Series 2, Vol. VIII.

Circled Surfaces.

For interpretation in circle geometry we shall consider the following ruled surfaces: general developables, developables whose generators are tangent to ϕ, ruled surfaces such that the S_3 determined by two consecutive generators is tangent to ϕ, general ruled surfaces (*i. e.*, surfaces such that the S_3 determined by two consecutive generators is not tangent to ϕ).

In S_4 there is one and only one line which cuts three given arbitrary lines.* Hence, on a ruled surface in space of four dimensions there is one curve which corresponds to the asymptotic lines in S_3; we will denote this curve by the term *asymptotic curve.* Two consecutive generators of a ruled surface will determine an S_3; in this way there are ∞^1 spaces of three dimensions determined by a ruled surface. Two such spaces S_3 infinitely near will intersect in a plane which contains the generator determining the S_3's and the tangent to the asymptotic curve at the point in which it cuts this generator. Hence, *on a circled surface there is one curve such that the circle C which cuts three consecutive generators of the surface in a point of this curve will also cut the same three generators again. Two consecutive generators of a circled surface determine an orthogonal complex of spheres, and two such consecutive complexes intersect in a congruence which is determined by C and the generator which it cuts.*

We saw that there was a projectivity between the points of a generator r of a ruled surfece and ∞^1 planes, forming a pencil, which pass through r. Considering the tangent plane as the plane pencil of lines tangent to the surface in a given point of r, we have: *The pencils of tangent circles which lie on the spheres of the pencil determined by a circle C of a circled surface, all have the circle C in common; that is, each sphere which passes through C is tangent to the surface in two points of C.* All the lines tangent to a ruled surface in S_4 along a generator r lie in an S_3. Hence, all the tangent circles touching the circled surface in points of a generator are orthogonal to a fixed sphere; and therefore the lines joining the two points of contact of the various spheres passing through C and tangent to the circled surface, must pass through a fixed point O, center of the fixed sphere. Now the following theorem can be stated: *Each sphere which passes through a circle C of a circled surface is tangent to the surface in two points; and the lines joining these two points of contact all pass through the same point O, and there is a* (1, 1) *correspondence between these lines and the spheres of the pencil.*†

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* Segre: *Rendi Conti di Palermo*, 1890.

† See Cosserat: "Sur le cercle considéré comme élément générateur de l'espace." Demartus: "Sur les surfaces à génératrice circulaire," *Annales de l'École Normale Supérieure*, 3rd series, Vol. II, p. 123.

In S_4 there is a quadric tangent to a ruled surface along a generator. Therefore in circled space there is a cyclide which is tangent along a circle generator of a circled surface and such that each circle of the cyclide cuts the given generator in two points; *i. e.*, has two-point contact with the circled surface. There are ∞^3 such cyclides.*

Enneper's classification of circled surfaces can be obtained here by considering the various relations which the S_3 determined by two consecutive generators can have with respect to the quadric ϕ.

Congruences of Circles.

Using homogeneous coordinates and letting

$$x_i = \phi_i(u, v) + t\psi_i(u, v), \qquad i = 1, 2, \dots, 5,$$

be the equations of a line of the congruence, then the equations of the line infinitely near to this one will be

$$x_i = \phi_i(u, v) + \frac{\partial\phi_i}{\partial u}du + \frac{\partial\phi_i}{\partial v}dv + t(\psi_i(u, v) + \frac{\partial\psi_i}{\partial u}du + \frac{\partial\psi_i}{\partial v}dv),$$

and the S_3 which these two lines determine is that determined by the points

$$\phi_i, \quad \psi_i, \quad \frac{\partial\phi_i}{\partial u}du + \frac{\partial\phi_i}{\partial v}dv, \quad \frac{\partial\psi_i}{\partial u}du + \frac{\partial\psi_i}{\partial v}dv;$$

that is,

$$\Sigma A_i x_i = 0,$$

where the A_i are determinants from the matrix

$$\left\| \phi_i, \quad \psi_i, \quad \frac{\partial\phi_i}{\partial u}du + \frac{\partial\phi_i}{\partial v}dv, \quad \frac{\partial\psi_i}{\partial u}du + \frac{\partial\psi_i}{\partial v}dv \right\|.$$

This hyperplane will be tangent to ϕ if

$$\Sigma A_i^2 = 0.$$

[Here it is supposed that the equation of ϕ has been put into the sum of squares.] The A_i are of degree two in du, dv; hence in each congruence of lines there are four ruled surfaces such that consecutive generators determine an S_3 tangent to ϕ. Therefore in circled space we have: *Through each circle of a congruence pass four circled surfaces such that each generator cuts the generator infinitely near to it in one point.*†

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* See Cosserat, *loc. cit.*

† See Darboux: "Théorie des Surfaces," Vol. II, p. 5. Cosserat, *loc. cit.*

We can take the point of tangency of the S to represent the S_3 itself, and in this way we have four surfaces of ϕ connected with each congruence. In ordinary space the points of these surfaces represent the point spheres through which pass the two circles infinitely near; that is, these surfaces of ϕ represent the focal surfaces of the congruence.

A congruence of spheres is represented in S_4 by a surface; and we saw that there were congruences of lines in S_4 composed of tangents to a surface along a system of curves, but such that the focal planes have no envelope. Hence, *a congruence of circles in which a circle intersects one infinitely near in two points is composed of the circle generators of a system of annular surfaces of a congruence of spheres.*

Now if the system of curves traced on the surface F in S_4 is a system of characteristics,* the tangents to these curves are also tangent to a second surface F' along a system of characteristics. (The tangents to any characteristic C of F touch F' along a characteristic C', but the lines are not tangent to C'.) Now the pencils of lines tangent to F' in two consecutive points of C' intersect in a line r tangent to C on F and to C_1 on F'. Then in S_3 the circle corresponding to such a line r is a circle of the annular surface corresponding to C and of the annular surface corresponding to C_1. Hence, *this circle is tangent to the focal surfaces of the congruences of spheres corresponding to F and F', and the focal surfaces of the congruence of circles corresponding to the lines tangent to F along the curves C are the four focal surfaces of the two congruences of spheres corresponding to F and F'.* The focal surfaces of any congruence of spheres can be made the focal surfaces of two such complexes of circles.

The two pencils of lines tangent to F' in two points infinitely near of C intersect two by two on the line r. Therefore in the corresponding congruence of spheres in S_3 the two focal points of any sphere and the two focal points of a sphere infinitely near, corresponding to two points infinitely near of a characteristic C', lie on ∞^1 spheres of a pencil and hence on a circle of an annular surface corresponding to a characteristic of F. The two surfaces F and F' may be interchanged. The circle L corresponding to such a line r is called a *principal circle* of the congruence of spheres.†

Then we see that, in general, *a system of principal circles of a congruence G of spheres is also a system of principal circles for another congruence G', and the*

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* Segre: "Su una classe di superficii, ecc.," *Atti di Torino*, 1907.

† Darboux: "Théorie des Surfaces," Vol. II.

envelope of such circles is composed of the four focal surfaces of the two congruences of spheres G and G'.

If the surface F in S_4 is of the parabolic type, the two systems of characteristics coincide and the surfaces F and F' coincide, and the two pencils of tangent lines in points infinitely near, of a characteristic, have a line in common; hence, in the congruence of spheres corresponding, the principal circles are tangent four times to the focal surface of the congruence of spheres.

A futher specialization occurs when the surface F in S_4 is a ruled surface. This corresponds to a congruence of spheres whose focal surface is a circled surface; that is, the congruence consists of ∞^1 pencils of spheres. The properties of such a congruence were discussed when circled surfaces were discussed. The principal circles of such a congruence are the circles of the circled surface each of which touches the focal surface in each point (*i. e.*, lies on the focal surface).

We saw that, in general, there were not more than two focal points on each line or more than two focal planes through each line, and that the two focal points must coincide when the focal planes coincided, but that it might happen that there was one focal point and an infinite number of focal planes, or vice versa. The conditions do not occur in general for each line of the congruence. In congruences of circles, likewise, it may happen that there is a single focal sphere (corresponding to focal point) and an infinite number of focal point pairs (corresponding to focal plane in S_4), or vice versa. An example of an infinite number of focal point pairs and a single focal sphere would be obtained if two congruences have an annular surface of principal spheres (spheres corresponding to a characteristic on a surface in S_4). Then the congruence of principal circles will have the circles of this annular surface as singular circles of the kind described above. As in the case of lines in S_4 this does not mean that an infinite number of annular surfaces pass through a circle of this congruence.

Triply Infinite Systems of Circles, C_3.

We saw that in a three-parameter system of lines in S_4 there were three developables passing through each line; hence, in a three-parameter system of circles three annular surfaces composed of circles of the system pass through each circle of the system.* The lines of the system in S_4 touched three hypersurfaces; hence, the circles of a three-parameter family are composed of the circles of ∞^2 annular surfaces contained in a complex of spheres. Such a system of annular surfaces is also contained in two other complexes. The triply infinite system of circles can then be generated by the intersection of three systems of ∞^5 circles composed of the circles on each sphere of a complex where it is cut by the spheres infinitely close.

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* See Cosserat, *loc. cit.*

We saw that a plane in S_4 corresponds to all the circles which pass through two points. We may therefore take this pair of points in S_3 as representing the plane in S_4. The planes which pass through a line l will correspond to the point pairs on the corresponding circle. The hyperplane tangent to a hypersurface ψ in a point P will correspond to a linear complex of spheres tangent to the complex of spheres which correspond to ψ. To a plane tangent to ψ in P will correspond a point pair on the sphere s which corresponds to P. Hence on s there are ∞^2 such point pairs, and the lines of these point pairs all pass through the same point (center of the fundamental sphere of the tangent linear complex). All planes tangent to ψ and passing through a line l will correspond to ∞^1 point pairs on the corresponding circle C. *The lines of these point pairs form a pencil.* In S_4 we saw that the focal planes which pass through a given line l of C_3 were tangent to the three focal surfaces. Let us call p_1, p_2, p_3 the three focal planes corresponding to the focal points P_1, P_2, P_3, and in circled space let p_1, p_2, p_3 be the point pairs corresponding to p_1, p_2, p_3, and s_1, s_2, s_3 be the spheres which correspond to P_1, P_2, P_3; and let ψ_1, ψ_2, ψ_3 be the complexes of spheres generated by s_1, s_2, s_3 respectively. The plane p_1 was seen to be tangent to the locus of P_2, P_3; hence, in circled space the ∞^1 point pairs on the circle C (corresponding to l) determined by the two complexes ψ_2 and ψ_3 have a point pair in common, but do not have a pair in common with the ∞^1 determined by ψ_1. If these three pencils of point pairs have a pair in common, the complexes would be such that ψ_1 and ψ_2, for example, would coincide; for in S_4 the focal plane p_1 would be tangent to all three hypersurfaces, and we saw that this could occur only if two of the focal hypersurfaces should coincide. If this condition should happen for each singular point pair (point pair corresponding to a focal plane), then all three of the focal complexes would coincide.

Singular Cases. Corresponding to the singular cases of lines in S_4 we have the following singular cases for circles: (1) Each point pair on any circle may be singular, but all corresponding to the same singular sphere; *e. g.*, all the circles which lie on a sphere. (2) There may be only one singular point pair, but each sphere passing through the circle being a singular sphere. (3) One sphere may be singular and ∞^1 point pairs of a pencil (*i. e.*, the lines of the point pairs form a pencil) correspond to it. (4) The singular spheres and point pairs may divide into two parts. In the first there is one singular sphere, but ∞^1 point pairs of a pencil correspond to it, while in the second there is one singular point pair, but each sphere is singular. (5) Each sphere is singular and there is a definite singular point pair corresponding to each sphere. These point pairs form a pencil. This differs from the preceding case in the fact that there is a (1, 1) correspondence between the singular spheres and the singular point pairs. An example of this kind of system is furnished by a complex of circles orthogonal to a fixed sphere.

Fourfold Infinite Systems of Circles, C_4.

The correlations between spheres and point pairs of a circle which belongs to a four-parameter family of circles form a linear system of three dimensions. Each sphere passing through the circle is a singular sphere and to each sphere corresponds a definite singular point pair.

In S_4 we saw that the singular planes generated a quadric cone having the given line of the system for vertex. To the singular plane in S_4 corresponds the singular point pair in circled space, and we shall now seek the envelope of the lines joining the singular point pairs corresponding to the spheres which pass through a given circle of the system.

The line in circled space which joins the points of a point pair (*i. e.*, the line joining the points through which pass all the circles correspond to the lines in S which lie in the same plane π) are represented in S_4 by the line in which π cuts the hyperplane t whose lines represent lines in circled space instead of circles. For through the two points must pass one line and this line must be represented by the intersection of t and π. The quadric cone of three dimensions mentioned above cuts t in an ordinary quadric cone of two dimensions. For the vertex of the three-dimensional cone cuts t in one point and the planes of this cone cut t in lines which must pass through this point and therefore generate a quadric cone in t. A quadric cone in t corresponds to a conic in ordinary space; hence: *Each sphere passing through a circle of a four-parameter family is a singular sphere, and to each sphere corresponds a definite singular point pair. The lines joining the points of these singular point pairs envelope a conic.*

It imposes one condition on the parameters if the above conic should degenerate. Hence, in a four-parameter family of circles there is a three-parameter family such that the correspondence between singular spheres and singular point pairs degenerates. Such circles may be called *singular* circles.

We saw that in a four-parameter family of lines in S_4 there where ∞^1 lines such that the correspondence between singular points and singular planes was a Chasles correlation; then in circled space there are ∞^1 circles in a four-parameter family such that the lines of the singular point pairs form a plane pencil.

Singular Cases. 1. There may be circles such that all the circled surfaces which belong to the system and pass through one of these circles have a tangent sphere and corresponding point pair in common. Such circles do not in general exist.

2. There may be a circle C for which the correspondence between the focal spheres and focal point pairs will degenerate into two parts: (1) One focal point pair which corresponds to each sphere passing through C; (2) a (1, 1) correspondence between the focal spheres and the focal point pairs. The lines of the point pairs in the second case generate a plane pencil.

3. There may be circles which behave like circles in a circled space formed by all the circles which cut a fixed sphere orthogonally. If this condition is an identity, we have the geometry of circles orthogonal to a fixed sphere.

Fivefold Infinite Systems of Circles, C_5.

The correlations between spheres and point pairs which belong to a fivefold infinite system of circles form a linear system of four dimensions.

Each sphere which passes through a given circle C of the system is a focal sphere, and to each focal sphere correspond ∞^1 focal point pairs. The lines of these point pairs generate a plane pencil and the ∞^1 pencils corresponding to the different spheres which pass through C have a line in common. There is a (1, 1) correspondence between the spheres passing through C and these pencils of lines.

In a five-parameter family of circles there is a four-parameter family such that the above correspondence between spheres and pencils of lines degenerates. In this case all the spheres passing through C correpond to the same pencil of lines. Also to each of a simply infinite number of point pairs on C corresponds each sphere passing through C. The lines of these point pairs form a plane pencil.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, *April*, 1909.